Journal of Approximation Theory **94**, 191–202 (1998)
Article No. AT983191

Local Convergence of Fourier Series with Respect to Periodized Wavelets

M. A. Skopina

St. Petersburg State University, Bibliotechnaja pl.-2, PM-PU, St. Petersburg 198904, Russia
E-mail: skopina@sk.usr.lgu.spb.su

Communicated by Ronald A. DeVore

Received June 3, 1996; accepted in revised form June 26, 1997

Fourier series with respect to wavelet orthonormal bases in $L_2([0, 1]^m)$ for a wide class of multiresolution analyses are studied. Convergence at Lebesgue and strong Lebesgue points is investigated and sufficient conditions for a.e. convergence are found. In addition, similar conditions for a.e. convergence of wavelet expansions of non-periodic functions are obtained. The latter essentially improve known results.

1. INTRODUCTION AND DEFINITIONS

A periodized wavelet basis of multiresolution analysis with sufficiently fast decaying mother- and father-wavelets is an orthonormal basis in $L_2([0, 1])$. Fourier series with respect to this basis (we call them wavelet Fourier series) have a remarkable property: uniform convergence of wavelet Fourier series of f to f for each continuous f (see, e.g., [1, Chap. 9]). It is of interest also to investigate the convergence of wavelet Fourier series of summable periodic functions at an individual point and almost everywhere. These problems were studied earlier for some special cases. The convergence of Fourier–Haar series at each Lebesgue point is well known (see, e.g., [2]). It is proved in [3] that 2^jth partial sums of Fourier series with respect to periodic spline wavelets converge art each Lebesgue point. We consider a similar problem for a wide class of wavelets in the multi-dimensional case. We will investigate the behavior of wavelet Fourier series of $f \in L([0, 1]^m)$ at points of two types: Lebesgue and so-called strong Lebesgue points (the latter were introduced by E. S. Belinskii [4]). One says that x is a Lebesgue point of f if

$$\lim_{h \to +0} \frac{1}{h^m} \int_{[-h, h]^m} |f(x+t) - f(x)| \, dt = 0.$$

0021-9045/98 $25.00

One says that x is a strong Lebesgue point of f if

$$\lim_{h_1, ..., h_m \to 0} \frac{1}{h_1 \cdots h_m} \int_{-h_1}^{h_1} dt_1 \cdots \int_{-h_m}^{h_m} dt_m\, |f(x+t) - f(x)| = 0,$$

$$\sup_{h_1, ..., h_m \neq 0} \frac{1}{h_1 \cdots h_m} \int_{-h_1}^{h_1} dt_1 \cdots \int_{-h_m}^{h_m} dt_m\, |f(x+t) - f(x)| < \infty.$$

It is well known that almost all points are Lebesgue points of $f \in L([0, 1]^m)$. It follows from results of S. Saks [6] that almost all points are strong Lebesgue points of $f \in L \log L([0, 1]^m)$, a fortiori of $f \in L_p$, $p > 1$. We consider an m-dimensional multiresolution analysis which is the tensor product of m given one-dimensional multiresolution analyses with father-wavelet φ and mother-wavelets ψ. In order to construct an orthonormal system in $L_2([0, 1]^m)$, generated by this analysis, it is necessary to impose some restrictions on φ and ψ. Without any other assumptions we prove that the wavelet Fourier series of $f \in L([0, 1]^m)$ converges at each strong Lebesgue point. Moreover, a close to best possible sufficient condition on φ and ψ for convergence at each Lebesgue point is found. This condition implies almost everywhere convergence for all $f \in L([0, 1]^m$. For $m = 2$ we have a sharper condition under which almost everywhere convergence holds for all $f \in L([0, 1]^2)$.

S. Kelly, M. Kon, and L. Raphael [3] investigated a similar problem for the non-periodic case. They showed that wavelet expansions converge almost everywhere for all $f \in L_p(R^m)$, $1 \leqslant p \leqslant \infty$, whenever mother- and father-wavelets of m-dimensional multiresolution analysis are bounded by a radial decreasing L_1 function. We improve this result for the cases when $p > 1$ or $m = 2$, $1 \leqslant p \leqslant \infty$. Our technique for periodic and non-periodic cases is similar to that of [3]: study bounds on summation kernels. We use two types of the above estimations for kernels: via a radial majorant of m-dimensional mother- and father-wavelets (this estimation was proved and used in [3]) and via a majorant of mother- and father-wavelets of one-dimensional multiresolution analysis generating m-dimensional wavelets.

Let us use the notations $\mathbf{T}^m = [-\frac{1}{2}, \frac{1}{2}]^m$; $E = \{1, ..., m\}$; $\mathscr{E} = \{e \subset E, e \neq E\}$; $g^e(x) = g^e(\varphi, \psi, x) = \prod_{l \in e} \varphi(x_l) \prod_{l \in E \setminus e} \psi(x_l)$, $x = (x_1, ..., x_m) \in \mathbf{R}^m$, $e \subset E$; $\hat{f}(x) = \int_{\mathbf{R}^m} f(u)\, e^{-2\pi i x \cdot u}\, du$, the Fourier transform of $f \in L(\mathbf{R}^m)$; $\sum_{k \in \mathbf{Z}^m} \hat{f}_k e^{2\pi i k \cdot x}$, the trigonometric Fourier series of $f \in L(\mathbf{T}^m)$.

Let φ and ψ be respectively the father- and mother-wavelets of one-dimensional multiresolution analysis

$$\cdots V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

We define spaces $\mathbf{V}_j$ by

$$\mathbf{V}_0 = V_0 \otimes \cdots \otimes V_0 = \overline{\mathrm{span}\left\{F(x),\, x \in \mathbf{R}^m,\, F(x) = \prod_{l=1}^{m} f_l(x_l),\, f_l \in V_0\right\}},$$

$$F \in \mathbf{V}_j \Leftrightarrow F(2^{-j}\cdot) \in \mathbf{V}_0, \qquad j \in \mathbf{Z}.$$

It is not difficult to show (see, e.g., [1, Chap. 10]) that the ladder of spaces $\mathbf{V}_j$, $j \in \mathbf{Z}$, forms a multiresolution analysis in $L_2(\mathbf{R}^m)$, the functions $\Phi_{j,n}(x) = 2^{mj/2} g^E(2^j x - n)$, $n \in \mathbf{Z}^m$, $x \in \mathbf{R}^m$, constitute an orthonormal basis in $\mathbf{V}_j$, and the functions $\Psi^e_{j,n}(x) = 2^{mj/2} g^e(2^j x - n)$, $n \in \mathbf{Z}^m$, $x \in \mathbf{R}^m$, $e \in \mathscr{E}$ constitute an orthonormal basis in the complement spaces $\mathbf{W}_j = \mathbf{V}_j \ominus \mathbf{V}_{j-1}$.

Let us assume that

$$|\varphi(t)|,\ |\psi(t)| \leqslant v(|t|), \tag{1}$$

where v is a monotone decreasing function which is summable on $[0, \infty)$. Define 1-periodic in each variable functions $T^E_{jn}(x) = \sum_{l \in \mathbf{Z}^m} \Phi_{jn}(x - l)$, $T^e_{jn}(x) = \sum_{l \in \mathbf{Z}^m} \Psi^e_{jn}(x - l)$. As for the one-dimensional case (see, e.g., [1, Chap. 9]), it is not difficult to verify that the spaces

$$\mathbf{V}^{per}_j = \overline{\mathrm{span}\{T^E_{jn},\, n \in \mathbf{Z}^m \cap [0, 2^j)^m\}}, \qquad j = 0, 1, ...,$$

form a ladder of multiresolution spaces in $L_2([0,1]^m)$ such that $\bigcup_{j=0}^{\infty} \mathbf{V}^{per}_j = L_2([0,1]^m)$, and the spaces

$$\mathbf{W}^{per}_j = \overline{\mathrm{span}\{T^E_{jn},\, n \in \mathbf{Z}^m \cap [0, 2^j)^m,\, e \in \mathscr{E}\}}, \qquad j = 0, 1, ...,$$

are orthogonal complements of $\mathbf{V}^{per}_j$ in $\mathbf{V}^{per}_{j+1}$. Thus, the functions T^E_{00}, T^e_{jn}, $e \in \mathscr{E}$, $j = 1, 2, ...$, $n \in \mathbf{Z}^m \cap [0, 2^j)^m$ constitute an orthogonal wavelet basis in $L_2([0,1])^m)$. Let $(f, T^e_{jn}) = \int_{[0,1]^m} f(x)\, T^e_{jn}(x)\, dx$ be Fourier coefficients with respect to this basis. For each positive integer j and the set $\Omega = \{\Omega^e \colon \Omega^e \subset [0, 2^j)^m\}_{e \in \mathscr{E}}$, we define partial sums

$$S_{j\Omega}(f) = (f, T^E_{00})\, T^E_{00} + \sum_{e \in \mathscr{E}} \sum_{i=0}^{j-1} \sum_{n \in \mathbf{Z}^m \cap [0, 2^j)^m} (f, T^e_{in})\, T^e_{in}$$

$$+ \sum_{e \in \mathscr{E}} \sum_{n \in \mathbf{Z}^m \cap \Omega^e} (f, T^e_{jn})\, T^e_{jn}.$$

We will say that a wavelet Fourier series converges if $S_{j\Omega_j}(f)$ converges, as $j \to \infty$, uniformly over all sequences $\{\Omega_j\}_j$.

2. MAIN RESULTS

THEOREM 1. *If φ and ψ are continuous functions satisfying* (1), *then the wavelet Fourier series of $f \in L([0,1]^m)$ converges to f at each strong Lebesgue point of f. In particular, for $f \in L \log L([0,1])^m)$ convergence holds almost everywhere.*

THEOREM 2. *If there exists a function v which is decreasing on $[0, \infty]$ and satisfies*

$$\int_0^\infty \rho^{m-1} v(\rho)\, d\rho < \infty, \tag{2}$$

and if φ and ψ are continuous functions satisfying (1), *then the wavelet Fourier series of $f \in L([0,1]^m)$ converges to f at each Lebesgue point of f. In particular, convergence holds almost everywhere.*

The condition on the functions φ, ψ in this theorem is close to sharp due to the following:

Remark. Relations (1), (2) in Theorem 2 cannot be replaced by $|\varphi(t)|$, $|\psi(t)| \ll t^{-m}$.

THEOREM 3. *If there exists a decreasing function v on $[0, \infty)$ and $\alpha > 0$ such that*

$$\int_1^\infty (\log \rho)^{1+\alpha}\, v(\rho)\, d\rho < \infty, \tag{3}$$

and if φ and ψ satisfy (1), *then the wavelet Fourier series of $f \in L([0,1]^2)$ converges to f almost everywhere.*

3. AUXILIARY RESULTS

LEMMA 1.

$$S_{j\Omega}(f, x) = 2^{mj} \int_{\mathbf{R}^m} f(t) \sum_{e \subset E} \sum_{r \in \Omega^e} \sum_{k \in \mathbf{Z}^m} g^e(2^j(t+k)+r)\, g^e(2^j(x+k)+r)\, dt,$$

where $\Omega^E := [0, 2^j)^m$.

Proof. First we prove that

$$(f, T_{00}^{E})\, T_{00}^{E}(x) + \sum_{e \in \mathscr{E}} \sum_{i=0}^{j-1} \sum_{r \in [0, 2^i)^m} (f, T_{ir}^{e})\, T_{ir}^{e}(x) = \sum_{r \in [0, 2^j)^m} (f, T_{jr}^{E})\, T_{jr}^{E}(x). \tag{4}$$

If $f \in L_2([0, 1]^m)$, then (4) is evident because the left and right expressions in (4) are the expansions of the projection of f onto $\mathbf{V}_j^{per}$ over two different orthonormal bases. Taking into account that, due to (1), the functions T_{ir}^{e} are bounded and noting that we can approximate an arbitrary $f \in L([0, 1]^m)$ by functions from $L_2([0, 1]^m)$ in L we prove (4) for f. So, we can write

$$\begin{aligned} S_{j\Omega}(f, x) &= \sum_{e \subset E} \sum_{r \in \Omega^e} (f, T_{jr}^{e})\, T_{jr}^{e}(x) \\ &= 2^{mj} \sum_{e \subset E} \sum_{r \in \Omega^e} \int_{\mathbf{T}^m} f(t) \sum_{k \in \mathbf{Z}^m} g^e(2^j(t+k)+r)\, dt \sum_{l \in \mathbf{Z}^m} g^e(2^j(x+l)+r) \\ &= 2^{mj} \sum_{e \subset E} \sum_{r \in \Omega^e} \sum_{l \in \mathbf{Z}^m} \int_{\mathbf{R}^m} f(t)\, g^e(2^j t+r)\, g^e(2^j(x+l)+r)\, dt. \end{aligned}$$

It remains only to change variables in the integral and change the order of summation and integration.

LEMMA 2. *Let v be an even monotone decreasing function which is summable on $[0, +\infty)$. Then, there holds*

$$\sum_{k \in \mathbf{Z}^m} \prod_{l=1}^{m} v(u_l + k_l)\, v(v_l + k_l) \leqslant A \prod_{l=1}^{m} v\left(\frac{u_l - v_l}{5}\right), \qquad u, v \in \mathbf{R}^m, \tag{5}$$

where A is a constant depending only on v and m.

Proof. First we note that since the left and right expressions in (5) are invariant under the translation $(u, v) \to (u+l, v+l)$ for $l \in \mathbf{Z}^m$, we can assume that $|u_l| \leqslant 1$, $l = 1, ..., m$. Let e_v be a subset of E such that $|v_l| > 4$ for $l \in e_v$ and $|v_l| \leqslant 4$ for $l \in E \setminus e_v$. For each $e \subset E$ we denote by $d(e)$ the set of $k \in \mathbf{Z}^m$ such that $|k_l| > |v_l|/2$ for $l \in e$ and $|k_l| \leqslant |v_l|/2$ for $l \in E \setminus e$. The left hand expression in (5) may be represented as the sum

$$\sum_{e \subset E} \sum_{k \in d(e)} \prod_{l=1}^{m} v(u_l + k_l)\, v(v_l + k_l).$$

It is enough to fix e and estimate the inner sum. Let $k \in d(e)$. If $l \in e_v \cap e$, then $|u_l + k_l| \geqslant |k_l|/2 \geqslant |v_l|/4$ and hence $v(u_l + k_l) \leqslant v(v_l/4) \leqslant v((u_l - v_l)/5)$;

if $l \in e_v \setminus e$, then $|v_l + k_l| \geqslant |v_l| - |k_l| \geqslant |v_l|/2$ and hence $\nu(u_l + k_l) \leqslant \nu(v_l/2) \leqslant \nu((u_l - v_l)/5)$. Applying these inequalities for $l \in e_v$ and replacing $\nu(u_l + k_l)$ or $\nu(v_l + k_l)$ by $\nu(0)$ for $l \in E \setminus e_v$, we have

$$\prod_{l=1}^{m} \nu(u_l + k_l)\, \nu(v_l + k_l)$$

$$\leqslant \prod_{l \in e_v} \nu\left(\frac{u_l - v_l}{5}\right) \prod_{l \in E \setminus e_v} \nu(0) \prod_{l \in e} \nu(v_l + k_l) \prod_{l \in E \setminus e} \nu(u_l + k_l).$$

To prove (5) it remains only to take into account that $\nu(u_l - v_l) \geqslant C\nu(0)$ for $l \in E \setminus e_v$ and that the sum $\sum_{k \in \mathbf{Z}} \nu(t-k)$ is bounded uniformly over $t \in \mathbf{R}$.

Lemma 3 [3]. *Let ν be an even monotone decreasing function which is summable on* $[0, +\infty)$ *and satisfies* (2). *Then, there holds*

$$\sum_{k \in \mathbf{Z}^m} \prod_{l=1}^{m} \nu(u_l + k_l)\, \nu(v_l + k_l) \leqslant B\nu\left(\frac{|u-v|}{5}\right), \qquad u, v \in \mathbf{R}^m,$$

where B is a constant depending only on ν and m.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Let x be a strong Lebesgue point of $f \in L[0,1]^m$. Integer translates of the mother-function φ constitute an orthonormal basis in the space V_0; this implies $\sum_{k \in \mathbf{Z}} \hat{\varphi}^2(t+k) = 1$ $(t \in \mathbf{R})$ (see [1, p. 132]). Moreover, due to (1), $\hat{\varphi}$ is bounded and continuous at 0. It follows that $\hat{\varphi}(0) = 1$ (see [1, p. 144]). On the basis of these equalities, we have $\int_0^1 e^{2\pi i k t} \sum_{l \in \mathbf{Z}} \varphi(t+l)\, dt = \hat{\varphi}(k) = \delta_{0k}$ and hence $T_{00}^{E}(x) = \prod_{i=1}^{m} \sum_{l_i \in \mathbf{Z}} \varphi(x_i + l_i) = 1$. Thus, the sums $S_{j\Omega}$ leave unchanged all functions $f \equiv \text{const}$ and therefore without loss of generality we can assume that $f(x) = 0$. Put $\mathcal{N}(u, v) = \sum_{l \in \mathbf{Z}^m} \prod_{s=1}^{m} \nu(u_s + l_s)\, \nu(v_s + l_s)$. By Lemma 1 and (1)

$$|S_{j\Omega}(f, x)| \leqslant 2^{m(j+1)} \int_{\mathbf{R}^m} |f(x+t)|\, \mathcal{N}(2^j(x+t), 2^j x)\, dt$$

$$= 2^m \sum_{k \in \mathbf{Z}^m} \int_{\mathbf{T}^m + k} |f(x + 2^{-j}t)|\, \mathcal{N}(2^j x + t, 2^j x)\, dt. \tag{6}$$

By the definition of a strong Lebesgue point, for any given $\varepsilon > 0$, we can choose $\delta > 0$, such that

$$\frac{1}{h_1 \cdots h_m} \int_{-h_1}^{h_1} \cdots \int_{-h_m}^{h_m} |f(x+t)| \, dt < \varepsilon, \qquad \text{for} \quad |h_i| < \delta, \, i = 1, ..., m,$$

$$\sup_{h_1, ..., h_m \neq 0} \frac{1}{h_1 \cdots h_m} \int_{-h_1}^{h_1} \cdots \int_{-h_m}^{h_m} |f(x+t)| \, dt = C < \infty.$$

Take a positive integer j_0 for which $2^{1-j_0/2} < \delta$ and $\int_{2^{-3+j_0/2}}^{\infty} v(\tau/5) \, d\tau < \varepsilon$. For $i \in \mathbf{Z}_+^m$ put $K(i) = \{k \in \mathbf{Z}^m; [2^{i_s - 1}] \leqslant |k_s| < 2^{i_s}, s = 1, ..., m\}$, where $[a]$ denotes the greatest integer in a. Due to (6) and Lemma 2, we have for all $j \geqslant j_0$

$$\begin{aligned} |S_{j\Omega}(f, x)| &\ll \sum_{i \in \mathbf{Z}_+^m} \prod_{s=1}^{m} v([2^{i_s - 2}]/5) \sum_{k \in K(i)} \int_{\mathbf{T}^m + k} |f(x + 2^{-j} t)| \, dt \\ &\leqslant \varepsilon \sum_{i \in \mathbf{Z}_+^m \cap j\mathbf{T}^m} \prod_{s=1}^{m} v([2^{i_s - 2}]/5) \, 2^{i_s + 1} \\ &\quad + C \sum_{i \in \mathbf{Z}_+^m \setminus j\mathbf{T}^m} \prod_{s=1}^{m} v([2^{i_s - 2}]/5) \, 2^{i_s + 1}. \end{aligned}$$

Applying the inequality $v([2^{i_s - 2}]/5) \, 2^{i_s + 1} \leqslant 16 \int_{2^{i_s - 3}}^{2^{i_s - 2}} v(\tau/5) \, d\tau$ for $i_s \geqslant 3$, we have

$$\begin{aligned} |S_{j\Omega}(f, x)| &\ll (v(0) + L)^m \, \varepsilon + C(v(0) + L)^{m-1} \\ &\quad \times \int_{2^{-3+j/2}}^{\infty} v(\tau/5) \, d\tau \leqslant (v(0) + L + C)(v(0) + L)^{m-1} \, \varepsilon, \end{aligned}$$

where $L = \int_0^\infty v(\tau) \, d\tau$. From this, we see that for each sequence $\{\Omega_j\}_{j=1}^{\infty}$ there holds $\lim_{j \to \infty} S_{j\Omega_j}(f, x) = 0$, and the convergence is uniform over all sequences $\{\Omega_j\}_{j=1}^{\infty}$.

Proof of Theorem 2. Let x be a Lebesgue point of $f \in L[0, 1]^m$. As in Theorem 1 we can assume that $f(x) = 0$. By the definition of a Lebesgue point, for each given $\varepsilon > 0$, we can find $\delta > 0$ such that $(1/h^m) \int_{h\mathbf{T}^m} |f(x)| \, dt < \varepsilon$ for all $h \in (0, \delta)$. Take j_0 for which $2^{-j_0/2} < \delta$ and $\int_{2^{-2+j/2}}^{\infty} \tau^{m-1} v(\tau/5) \, d\tau \leqslant \varepsilon$. Due to Lemma 3 and (6), we have for all $j \geqslant j_0$

$$\begin{aligned} |S_{j\Omega}(f, x)| &\ll v(0) \int_{\mathbf{T}^m} |f(x + 2^{-j} t)| \, dt \\ &\quad + \sum_{i=0}^{\infty} v(2^{i-1}/5) \int_{2^{i+1}\mathbf{T}^m \setminus 2^i \mathbf{T}^m} |f(x + 2^{-j} t)| \, dt \\ &\ll \varepsilon v(0) + \varepsilon \sum_{0 \leqslant i \leqslant j/2} 2^{im} v(2^{i-1}/5) + \|f\|_1 \sum_{i > j/2} 2^{im} v(2^{i-1}/5). \end{aligned}$$

Applying the inequality $2^{im}\nu(2^{i-1}/5) \leqslant 2^{2m}\int_{2^{i-2}}^{2^{i-1}} \tau^{m-1}\nu(\tau/5)\,d\tau$, we have

$$|S_{j\Omega}(f,x)| \ll \left[\nu(0)+\int_0^\infty \tau^{m-1}\nu(\tau)\,d\tau\right]\varepsilon + \|f\|_1 \int_{2^{-2+j/2}}^\infty \tau^{m-1}\nu(\tau/5)\,d\tau$$

$$\leqslant \left[\nu(0)+\int_0^\infty \tau^{m-1}\nu(\tau)\,d\tau + \|f\|_1\right]\varepsilon.$$

From this, we see that for each sequence $\{\Omega_j\}_{j=1}^\infty$ there holds $\lim_{j\to\infty} S_{j\Omega_j}(f,x)=0$, and the convergence is uniform over all sequences $\{\Omega_j\}_{j=1}^\infty$.

Proof of Remark. We restrict our focus to considering the case $m=2$. For arbitrary m, a similar construction proves the remark.

Define φ by

$$\hat{\varphi}(u) = \begin{cases} 1, & \text{if } |u| \leqslant 1/3, \\ \left|\sin \dfrac{3\pi u}{2}\right|, & \text{if } 1/3 \leqslant |u| \leqslant 2/3, \\ 0, & \text{if } |u| \geqslant 2/3. \end{cases}$$

One can recognize in φ the father-wavelet of Meyer's non-smooth multiresolution analysis [6]. We construct a tensor product of two such analyses and consider the corresponding wavelet Fourier series. It is not difficult to compute φ explicitly

$$\varphi(u) = \frac{\sin 2\pi u/3}{2\pi u} + \frac{6\cos 4\pi u/3 + 8u \sin 2\pi u/3}{\pi(9-16u^2)}.$$

We can also compute the father-wavelet ψ due to equation $\hat{\psi}(u) = e^{i\pi u}(\hat{\varphi}(u+1) + \hat{\varphi}(u-1))\,\hat{\varphi}(u/2)$ (see [1, p. 138]) and verify that $\psi(u) = O(u^{-2})$. We see from this that φ and ψ satisfy (1) and do not satisfy the condition of Theorem 2. We put $S_j(f,x) = S_{j\Omega_\varnothing}(f,x)$, where $\Omega_\varnothing = \{\Omega^e : \Omega^e = \varnothing\ \forall e \in \mathscr{E}\}$, and prove that there exists a function $f \in L(\mathbf{T}^2)$ such that the sequence $\{S_j(f)\}$ diverges at some Lebesgue point.

Take $f_n(x) = e^{2\pi i n\cdot x}$, $n \in \mathbf{Z}^2$. By Lemma 1

$$S_j(f_n,x) = 2^{2j}\int_{\mathbf{R}^2} f_n(t) \sum_{r\in[0,2^j)^2}\sum_{k\in\mathbf{Z}^2} g^E(2^j(t+k)+r)\, g^E(2^j(x+k)+r)\,dt$$

$$= \prod_{l=1}^{2}\sum_{k_l\in\mathbf{Z}} 2^j \int_{-\infty}^{\infty} e^{2\pi i n_l t_l}\varphi(2^j t_l - k_l)\,\varphi(2^j x_l - k_l)\,dt_l$$

$$= \prod_{l=1}^{2} \hat{\varphi}(2^{-j}n_l) \sum_{k_l\in\mathbf{Z}} e^{2\pi i n_l 2^{-j}k_l}\varphi(2^j x_l - k_l).$$

Applying the Poisson summation formula and taking into account that $\operatorname{supp} \varphi \cap \operatorname{supp} \varphi(\cdot + k) = 0$ for $k \in \mathbf{Z}$, $|k| \geqslant 2$, we have

$$S_j(f_n, x) = \prod_{l=1}^{2} (e^{2\pi i n_l x_l} \hat{\varphi}(2^{-j} n_l) + e^{2\pi i (n_l - 2^j) x_l} \hat{\varphi}(1 - 2^{-j} n_l) + e^{2\pi i (n_l + 2^j) x_l} \hat{\varphi}(1 + 2^{-j} n_l))\, \hat{\varphi}(2^{-j} n_l).$$

Hence, we can express $S_j(f, x)$ for arbitrary $f \in L(\mathbf{T}^2)$ via trigonometric Fourier coefficients

$$S_j(f, x) = \sum_{l=0}^{2} \sum_{k=0}^{2} \gamma_l(2^j x_l)\, \gamma_k(2^j x_2) \sum_{n \in \mathbf{Z}} \alpha_{lk}(2^{-j} n)\, \hat{f}_n e^{2\pi i n \cdot x}, \tag{7}$$

where $\gamma_0(u) = 1$, $\gamma_1(u) = 2 \cos 2\pi u$, $\gamma_2(u) = 2i \sin 2\pi u$, $\alpha_{lk}(u, v) = \alpha_l(u)\, \alpha_k(v)$, $\alpha_0(u) = \hat{\varphi}^2(u)$, $\alpha_1(u) = \hat{\varphi}(u)[\hat{\varphi}(1+u) + \hat{\varphi}(1-u)]$, $\alpha_2(u) = \hat{\varphi}(u)[\hat{\varphi}(1+u) - \hat{\varphi}(1-u)]$. The sums over n in (7) are linear summation methods of trigonometric Fourier series. We apply to them the following result of E. S. Belinskii [4]: a sequence of linear means $\sigma_R(f, x) = \sum_{n \in \mathbf{Z}^m} \alpha(R^{-1} n)\, \hat{f}_n e^{2\pi i n x}$ converges as $R \to \infty$ at each Lebesgue point of $f \in L([0, 1]^m)$ if and only if

$$\int_0^\infty \tau^{m-1} \sup_{|t| \geqslant \tau} |\hat{\alpha}(t)|\, d\tau < \infty. \tag{8}$$

The computations give

$$\hat{\alpha}_{00}(u, v) = \frac{81(\sin 2\pi u/3 + \sin 4\pi u/3)(\sin 2\pi v/3 + \sin 4\pi v/3)}{4\pi^2 uv(9 - 4u^2)(9 - 4v^2)},$$

$$\hat{\alpha}_{10}(v, u) = \hat{\alpha}_{01}(u, v) = \frac{27(\sin 2\pi u/3 + \sin 4\pi u/3)(\cos 2\pi v/3 + \cos 4\pi v/3)}{2\pi^2 u(9 - 4u^2)(9 - 4v^2)},$$

$$\hat{\alpha}_{11}(u, v) = \frac{9(\cos 2\pi u/3 + \cos 4\pi u/3)(\cos 2\pi v/3 + \cos 4\pi v/3)}{\pi^2(9 - 4u^2)(9 - 4v^2)}.$$

We see that (8) is fulfilled for α_{00}, and not fulfilled for $2\alpha_{11} + \alpha_{10} + \alpha_{01}$. Consequently, we can construct a function $f \in L([0, 1]^2)$ for which the point $(0, 0)$ belongs to its Lebesgue set and $S_j(f, 0, 0)$ divergences as $j \to \infty$.

Proof of Theorem 3. Let f in $L([0, 1]^2)$ and put

$$J_l(f, x, h, r) = \frac{2^l}{h^2} \int_{-h}^{h} dt_1 \int_{-2^{-l}h}^{2^{-l}h} |f(x + C_r t) - f(x)|\, dt_2,$$

where $h>0$, $r=1, 2$, $C_1=\binom{01}{10}$, $C_2=\binom{10}{01}$. It is proved in [7] that

$$\lim_{h\to 0}\ \sup_{r=1,2}\ \sup_{l\in\mathbf{Z}_+} (1+l)^{-1-\alpha} J_l(f, x, h, r)=0, \tag{9}$$

$$\sup_{h>0}\ \sup_{r=1,2}\ \sup_{l\in\mathbf{Z}_+} (1+l)^{-1-\alpha} J_l(f, x, h, r)=C<\infty \tag{10}$$

for almost all $x\in\mathbf{R}^2$. For any x, satisfying (9), (10), and $\varepsilon>0$, we can find $\delta>0$ such that

$$\sup_{r=1,2}\ \sup_{l\in\mathbf{Z}_+} (1+l)^{-1-\alpha} J_l(f, x, h, r)<\varepsilon \tag{11}$$

for all $h\in(0, \delta)$. As in Theorem 1 we can assume that $f(x)=0$. We take a positive integer j_0 for which $2^{1-j_0/2}<\delta$, $\int_{2^{-3+j_0/2}}^{\infty} v(\tau/5)\log^{1+\alpha}16\tau\,d\tau\leqslant\varepsilon$. Due to (6) and Lemma 2, we have for all $j\geqslant j_0$

$$\begin{aligned}|S_{j\Omega}(f, x)| &\ll \sum_{i\in\mathbf{Z}_+^2}\ \sum_{k\in K(i)} \int_{\mathbf{T}^2+k} |f(x+2^{-j}t)|\ v(|t_1|/5)\ v(|t_2|/5)\ dt\\ &\leqslant \sum_{i\in\mathbf{Z}_+^2} v([2^{i_1-2}]/5)\ v([2^{i_2-2}]/5) \int_{-2^{i_1+1}}^{2^{i_1+1}} dt_1\\ &\quad\times\int_{-2^{i_2+1}}^{2^{i_2+1}} |f(x+2^{-j}t)|\ dt_2.\end{aligned}$$

We use here the notations of Theorem 1. Due to symmetry, if suffices to estimate

$$\begin{aligned}&\sum_{i_1=0}^{\infty}\ \sum_{i_2=0}^{i_1} v([2^{i_1-2}]/5)\ v([2^{i_2-2}]/5) \int_{-2^{i_1+1}}^{2^{i_1+1}} dt_1 \int_{-2^{i_2+1}}^{2^{i_2+1}} |f(x+2^{-j}t)|\ dt_2\\ &\quad= \sum_{i_1=0}^{\infty}\ \sum_{i_2=0}^{i_1} v([2^{i_1-2}]/5)\ v([2^{i_2-2}]/5)\ 2^{i_1+i_2+2} J_{i_1-i_2}(f, x, 2^{i_1-j+1}, 1).\end{aligned}$$

We split the sum over i_1 into two parts: $i_1\leqslant j/2$ and $i_1>j/2$. Since $2^{i_1-j+1}\leqslant\delta$, for $i_1\leqslant j/2$, due to (11), the first part does not exceed

$$\begin{aligned}\varepsilon \sum_{0\leqslant i_1\leqslant j/2} 2^{i_1+5}\ v([2^{i_1-2}]/5) \Bigg[&(1+i_1)^{1+\alpha}\ v(0)\\ &+\sum_{i_2=2}^{i_1} (1+i_1-i_2)^{1+\alpha} \int_{2^{i_2-3}}^{2^{i_2-2}} v(\tau/5)\ d\tau\Bigg].\end{aligned} \tag{12}$$

Applying the Abel transformation to the sum over i_2, we have for $i_2 \geqslant 2$,

$$\sum_{i_2=2}^{i_1} (1+i_1-i_2)^{1+\alpha} \int_{2^{i_2-3}}^{2^{i_2-2}} v(\tau/5)\, d\tau$$

$$\leqslant \int_0^\infty v(\tau/5)\, d\tau \left[1 + \sum_{i_2=2}^{i_1} ((1+i_1-i_2)^{1+\alpha} - (i_1-i_2)^{1+\alpha}) \right]$$

$$\leqslant 2 \int_0^\infty v(\tau/5)\, d\tau (i_1-1)^{\alpha+1}.$$

Thus the sum (12) can be estimated from above by

$$2^9 \varepsilon \left(v(0) + \int_0^\infty v(\tau/5) \log^{1+\alpha} 16\tau\, d\tau \right) \left(v(0) + \int_0^\infty v(\tau/5)\, d\tau \right).$$

Similarly, using (10) instead of (11), we estimate the second part by

$$2^9 C \left(\int_{2^{-3+j/2}}^\infty v(\tau/5) \log^{1+\alpha} 16\tau\, d\tau \right) \left(v(0) + \int_0^\infty v(\tau/5)\, d\tau \right)$$

$$\leqslant 2^9 C \varepsilon \left(v(0) + \int_0^\infty v(\tau/5)\, d\tau \right).$$

From this, we see that for each sequence $\{\Omega_j\}_{j=1}^\infty$ the relation $\lim_{j\to\infty} S_{j\Omega_j}(f, x) = 0$ holds true and the convergence is uniform over all sequences $\{\Omega_j\}_{j=1}^\infty$.

5. NON-PERIODIC CASE

Following [3] we define the multiresolution expansion of $f \in L_p(\mathbf{R}^m)$, $0 \leqslant p \leqslant \infty$, by the sequence $\{P_j\}_{j\in\mathbf{Z}}$

$$P_j f(x) = \sum_{k\in\mathbf{Z}^m} 2^{mj} \left(\int_{\mathbf{R}^m} f(t)\, g^E(2^j t + k)\, dt \right) g^E(2^j x + k);$$

and the wavelet and scaling expansions of f by the series

$$\sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}^m} \sum_{e\in\mathscr{E}} 2^{mj} \left(\int_{\mathbf{R}^m} f(t)\, g^e(2^j t + k)\, dt \right) g^e(2^j x + k) \tag{13}$$

and

$$\sum_{k\in\mathbf{Z}^m}\left(\int_{\mathbf{R}^m} f(t)\, g^E(t+k)\,dt\right) g^E(x+k)$$
$$+\sum_{j\in\mathbf{Z}}\ \sum_{k\in\mathbf{Z}^m_+}\ \sum_{e\in\mathscr{E}} 2^{mj}\left(\int_{\mathbf{R}^m} f(t)\, g^e(2^jt+k)\,dt\right) g^e(2^jx+k), \qquad (14)$$

respectively. The following theorems improve some results of [3].

THEOREM 4. *If there exists a decreasing function* v *which is summable on* $[0, \infty)$ *and such that* $|\varphi(t)| \leqslant v(|t|)$, *then the multiresolution expansion of each* $f\in L_p(\mathbf{R}^m)$, $p>1$, *converges almost everywhere. If* (1) *holds, then the scaling and wavelet expansions of each* $f\in L_p(\mathbf{R}^m)$, $p>1$, *also converge almost everywhere.*

This theorem may be proved similarly to Theorem 1.

THEOREM 5. *If there exists a decreasing function* v *on* $[0, \infty]$ *such that* (3) *holds and* $|\varphi(t)| \leqslant v(|t|)$, *then the multiresolution expansion of each* $f\in L(\mathbf{R}^2)$ *converges almost everywhere. If, in addition, there exists a decreasing function* μ *which is summable on* $[0, \infty)$ *and such that* $|\psi(t)| \leqslant \mu(|t|)$, *then the scaling and wavelet expansions of any* $f\in L(\mathbf{R}^2)$ *also converge almost everywhere.*

This theorem may be proved similarly to Theorem 3.

REFERENCES

1. I. Daubechies, "Ten Lectures on Wavelets," CBMS-NSR Series in Appl. Math., SIAM, Philadelphia, 1992.
2. B. S. Kashin and A. A. Saakyan, "Orthogonal Series," Nauka, Moscow, 1984.
3. S. E. Kelly, M. A. Kon, and L. A. Raphael, Local convergence for wavelet expansions, preprint.
4. S. I. Belinskii, The summability of Fourier series at Lebesgue points, *Theory Funct. Functional Anal. Appl.* **23** (1971).
5. Y. Meyer, "Ondelettes," Hermann, Paris, 1990.
6. S. Saks, On the strong derivatives of function of intervals, *Fund. Math.* **25** (1935).
7. M. A. Skopina, The generalized Lebesgue sets of functions of two variables, *in* "Approximation Theory," Colloq. Math. Soc. János Bolyai, Vol. 58, Kecskemet, Hungary, 1990.